the absence of impurity atoms is

$$\eta_{\omega}^{-1}(\zeta_n) = 1 + 2\pi n_i N_{sd}(0)\,\frac{(m_s m_d)^{1/2}}{m_s + m_d} \times \frac{1}{\omega_n}\left(\langle V_{sd}^2\rangle_{\Omega} + \langle V_s V_d\rangle_{\Omega}\right) . \quad (45)$$

Using this result in (42), we see that an additional integral occurs in the self-consistent equation at T equal the old critical temperature. The effect of this additional term is that the gap energy at the old critical temperature is not zero. Whether this change in the gap energy is positive or negative, resulting in an increased or decreased critical temperature, respectively, will depend on the same consideration as discussed by Chow in the interband phonon-coupling limit.

V. CONCLUSION

As we stated in Sec. IV, the exact behavior of the critical temperature can be obtained by solving Eqs. (40)–(42) along with (32)–(36). However, this was not attempted due to the mathematical difficulties involved in attempting to do it except in two limiting cases. It is hoped, however, that by picking the right values for the impurity scattering potentials $V_s(p)$, $V_d(p)$, and $V_{sd}(p)$ the defining set of equations [(32)–(36)] will become simple enough for the substitution (43) to be useful.

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Parameter $\kappa_2(T)$ for Strong Coupling Superconducting Alloys*

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The parameter $\kappa_2(T)$ is studied for superconducting alloys (dirty limit) in the electron-phonon model. An equation for $\kappa_2(T)$ is derived which contains the frequency-dependent part of the order parameter right at the transition from the normal to the superconducting state and which can be used as a starting point for a detailed numerical calculation. At T_c, κ_2 is expressed in terms of other measurable quantities, and at $T=0$ a rough estimate of the order of magnitude of the strong coupling corrections to κ_{2BCS} is given. It is found that the strong coupling corrections to κ_2 are very small, even for lead.

I. INTRODUCTION

In the generalization of the Ginzburg-Landau (GL) theory[1] to lower temperatures,[2,3] the well-known GL parameter κ is replaced by three different temperature-dependent parameters $\kappa_i(t)$, where $i = 1$ to 3, and $t = T/T_c$ = reduced temperature. Within the framework of the weak coupling microscopic theory, it was shown[2] that for $t \to 1$, all $\kappa_i(t) \to \kappa$. For lower temperatures, Eilenberger[4] has made a detailed calculation of $\kappa_1(t)$ and $\kappa_2(t)$ and studied in particular the influence of the mean free path on $\kappa_1(t)$ and $\kappa_2(t)$. In the extreme dirty limit, Eilenberger confirmed the result found earlier by Caroli *et al.*,[3] that $\kappa_1(0) = \kappa_2(0)$. All these calculations, however, apply to the weak coupling limit only.

For strong coupling superconductors, Eilenberger and Ambegaokar[5] (EA) presented a calculation of $H_{c2}(t)$ in the region $(1-t) \ll 1$. An extension of this theory can be combined with the well-known $H_c(t)$ calculations to derive $\kappa_1(t)$. Such a calcula-

tion of $\kappa_1(t)$, however, requires a large numerical effort which has not yet been done. Experimentally, there is much evidence [6-8] that the temperature dependence of $\kappa_1(t)/\kappa_1(1)$ is stronger than that predicted by weak coupling theory. On the other hand, experiments[7] indicate that $\kappa_2(t)/\kappa_2(1)$ is much less affected by strong coupling than $\kappa_1(t)/\kappa_1(1)$. Therefore, it may even be possible that the BCS relation $\kappa_1(t) \leqq \kappa_2(t)$ is changed into $\kappa_1(t) \geqq \kappa_2(t)$ for certain temperatures and certain values of the electron mean free path.

In this paper we study the parameter $\kappa_2(t)$, restricting ourselves to the extreme dirty limit. Following closely the work of EA, we derive an equation for $\kappa_2(t)$ which contains the frequency-dependent part of the order parameter for the H_{c2} solution, thus showing that from the numerical solution of the H_{c2} problem it is easy to get also the parameter $\kappa_2(t)$. But we will not undertake the large numerical effort which is necessary to get explicit expressions for both $\kappa_1(t)$ and $\kappa_2(t)$. We will only give a rough estimate of the deviations of $\kappa_2(t=0)$ from the well-known BCS value, and we will express $\kappa_2(1)$ in terms of measurable quantities. As a by-product, we derive $\kappa_1(1) = \kappa_2(1)$. This equality is expected from the GL theory, and one can show with methods used in this paper that its validity is not restricted to dirty materials. From a mathematical point of view, the relation $\kappa_1(1) = \kappa_2(1)$ is due to a certain separability of the frequency-dependent order parameter just below the transition into the superconducting state.

II. EQUATION FOR $\kappa_2(T)$

Our starting point is the Green's-function matrix in the phonon model given by EA, which we expand into powers of the frequency-dependent order parameter $\Phi(\vec{r}, \omega)$. The Green's function in the normal state is given by

$$[i\omega - (1/2m)(i^{-1}\vec{\delta} + e\vec{A})^2 - V(\vec{r}) - \Sigma_N(\omega, \vec{r})] \times G_N(\omega, \vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') \ , \tag{1}$$

with a self-energy determined by

$$\Sigma_N(\omega, \vec{r}) = -T \sum_{\omega'} \lambda(\omega, \omega')\, G_N(\omega', \vec{r}, \vec{r}) \ . \tag{2}$$

The symbols in these equations have their standard meanings and we work in units with $c = \hbar = k_B = 1$. The normal Green's function in the superconducting state expanded to second order in $\Phi(\omega, \vec{r})$ is given by

$$G(\omega, \vec{r}, \vec{r}') = G_N(\omega, \vec{r}, \vec{r}') + G^{(2)}(\omega, \vec{r}, \vec{r}') - \int d1\, d2\, G_N(\omega, \vec{r}, 1)\, \Phi(\omega, 1)\, G_N^*(\omega, 1, 2) \times \Phi^*(\omega, 2)\, G_N(\omega, 2, \vec{r}') \ . \tag{3}$$

The asterisk denotes complex conjugation. The quantity $G^{(2)}$ is due to the modified electron self-energy in the superconducting state for strong coupling materials, and it is given by

$$G^{(2)}(\omega, \vec{r}, \vec{r}') = \int d1\, G_N(\omega, \vec{r}, 1)\, \Sigma^{(2)}(\omega, 1)\, G_N(\omega, 1, \vec{r}') \ , \tag{4}$$

while $\Sigma^{(2)}$ has to be determined self-consistently according to

$$\Sigma^{(2)}(\omega, \vec{r}) = -T \sum_{\omega'} \lambda(\omega, \omega')[G(\omega', r, r) - G_N(\omega', r, r)] . \tag{5}$$

Finally, we have to write the equation for the order parameter $\Phi(\omega, r)$:

$$\Phi(\omega', \vec{r}) = -T \sum\nolimits_{\omega} \lambda(\omega', \omega) \times \{ \int d1\, G_N(\omega, \vec{r}, 1)\, \Phi(\omega, 1) \times G_N^*(\omega, 1, \vec{r}) + R^{(3)}(\omega, \vec{r}) \} , \tag{6}$$

with an $R^{(3)}$ given by

$$\begin{aligned} R^{(3)}(\omega, \vec{r}) &= G_N(\omega, \vec{r}, 1)\, \Phi(\omega, 1) \\ &\times G_N^*(\omega, 1, 2)\, \Sigma^{(2)*}(\omega, 2)\, G_N^*(\omega, 2, \vec{r}) \\ &+ G_N(\omega, \vec{r}, 1)\, \Sigma^{(2)}(\omega, 1)\, G_N(\omega, 1, 2)\, \Phi(\omega, 2) \\ &\times G_N^*(\omega, 2, \vec{r}) - G_N(\omega, \vec{r}, 1) \Phi(\omega, 1) G_N^*(\omega, 1, 2) \\ &\times \Phi^*(\omega, 2)\, G_N(\omega, 2, 3)\, \Phi(\omega, 3)\, G_N^*(\omega, 3, \vec{r}) . \end{aligned} \tag{7}$$

Of course, in all the above equations we have to perform the impurity average in an appropriate way. In doing this, we use the usual approximation, neglecting all overcrossing diagrams, i.e., we neglect those diagrams where an impurity line connects a Green's function with Φ or Σ.

First of all, we note that the self-energy $\Sigma_N \times (\omega, \vec{r})$ is independent of the magnetic field and the impurity scattering (after the impurity average) and thus independent of $\vec{r}$. Putting $\Sigma_N(\omega) = i\omega \times [1 - Z(\omega)]$, one gets

$$\omega[1 - Z(\omega)] = \pi N(0)\, T \sum_{\omega'} \lambda(\omega, \omega')\, \mathrm{sgn}\,\omega' . \tag{8}$$

Therefore, the renormalization of the normal-state Green's functions due to the electron-phonon interaction can easily be taken into account by the replacement of $i\omega$ by $i\omega\, Z(\omega)$ in the weak coupling Green's functions. Bearing this in mind, we see that the above formulas differ from the corresponding weak coupling expressions mainly in the following two ways: (a) The order parameter depends on ω, and (b) the second term on the right-hand side of Eq. (3) and the first and second term on the right-hand side of Eq. (7) are absent in the weak coupling case. But as we will see in the following, the calculation of $\kappa_2(T)$ can be done very

similarly to the weak coupling calculation.

Let us consider the transition into the superconducting state which is determined by the linear part of Eq. (6). Performing the impurity average and considering only the dirty limit, one gets

$$\Phi^{(0)}(\omega,\vec{r}) = -T\sum\lambda(\omega,\omega')\times\{2\pi N(0)/[2|\omega' Z(\omega')| - D\tilde{\vec{\delta}}^2]\}\Phi^{(0)}(\omega',\vec{r}) \ . \quad (9)$$

The operator $\tilde{\vec{\delta}}$ is defined by

$$\tilde{\vec{\delta}} = \frac{\partial}{\partial\vec{r}} + 2ie\,A(\vec{r}) \ , \quad (10)$$

and $A(\vec{r})$ denotes a vector potential of the external magnetic field, while $D = vl/3$ denotes the diffusion constant. Equation (9) clearly allows a separation of variables. We put

$$\Phi^{(0)}(\omega,\vec{r}) = \chi(\omega)\,\Psi(\vec{r}) \quad (11)$$

and choose Ψ as an eigenfunction of $\tilde{\vec{\delta}}^2$ to obtain

$$\tilde{\vec{\delta}}^2\Psi(\vec{r}) = -\epsilon_0\Psi(\vec{r}) \ . \quad (12)$$

Then we get

$$\chi(\omega) = -T\sum_{\omega'}\lambda(\omega,\omega')\frac{2\pi N}{2|\omega' Z(\omega')| + D\epsilon_0}\chi(\omega') \ . \quad (13)$$

These equations determine the upper critical field $2eH_{c2} = \epsilon_0$.

For an external magnetic field H_e slightly smaller than H_{c2} (and T fixed), we choose the ansatz

$$\Phi(\omega,\vec{r}) = \alpha\Psi(\vec{r})\chi(\omega) + \alpha^3\varphi(\vec{r},\omega) + \cdots \ . \quad (14)$$

$\chi(\omega)$, which can be assumed to be real, is a normalized solution of Eq. (13) and is independent of the magnetic field. $\Psi(\vec{r})$ is now understood to be the normalized ground-state solution[9] of Eq. (12) in the external magnetic field $H_e < H_{c2}$, which shows the periodicity of the fluxoid lattice, and α is an expansion parameter proportional to $(H_{c2} - H_e)^{1/2}$. Our aim is to calculate this quantity α. We multiply Eq. (7) by

$$2\pi N(0)\,T\chi(\omega')\,\alpha\Psi^*(r)/[2|\omega' Z(\omega')| + D\epsilon_0] \ ,$$

integrate over a unit cell of the fluxoid lattice, and sum over ω'. Using the symmetry property $\lambda(\omega,\omega') = \lambda(\omega',\omega)$ of the kernel λ, we get

$$\sum_\omega \frac{2\pi N(0)T}{2|\omega Z(\omega)| + D\epsilon_0}\chi(\omega)\alpha\int\Psi^*(\vec{r})\Phi(\omega,\vec{r})\,d^3r$$

$$= T\sum_\omega \alpha\chi(\omega)\Big[\int d^3r\int d1\ \Psi^*(\vec{r})\langle G_N(\omega,\vec{r},1)G_N^*(\omega,1,\vec{r})\rangle\Phi(\omega,1) + \int d^3r\int d1\ \Psi^*(\vec{r})\langle R^{(3)}(\omega,\vec{r})\rangle\Big] \ , \quad (15)$$

where $\langle\cdots\rangle$ denotes the impurity average. We use the well-known expression

$$\langle G_N(\omega,\vec{r},\vec{r}')G_N^*(\omega,\vec{r}',\vec{r})\rangle = \frac{2\pi N(0)}{2|\omega Z(\omega)| - D(\tilde{\vec{\delta}}^2 + 2ie\tilde{\vec{\delta}}\cdot\vec{A}_s + 2ie\vec{A}_s\cdot\tilde{\vec{\delta}} - 4e^2\vec{A}_s^2)}\delta(r,r') \ , \quad (16)$$

where $\vec{A}_s$ is a vector potential of the supercurrents (which can be gauged in such a way that it becomes proportional to α^2), insert Eq. (14) into Eq. (15), and take only terms to the order of α^3 into account. Then we see immediately that the function $\varphi(\vec{r},\omega)$ cancels out and we are left with the following expression:

$$B\left((\epsilon_e - \epsilon_0) - \frac{2ie(\Psi|\vec{A}_s\cdot\tilde{\vec{\delta}} + \tilde{\vec{\delta}}\cdot\vec{A}_s|\Psi)}{(\Psi|\Psi)}\right) = T\sum_\omega\alpha^2[\chi(\omega)]^2\bar{R}^{(3)}(\omega) \ , \quad (17)$$

where we have defined

$$B = DT\alpha^2 2\pi N(0)\sum_\omega[\chi(\omega)]^2/[2|\omega Z(\omega)| + D\epsilon_0]^2 \quad (18)$$

and

$$R^{(3)}(\omega)\frac{1}{(\Psi|\Psi)}\int d^3r\int d1\,d2\,\{\langle\Sigma^{(2)*}(\omega,2)\rangle\Psi^*(\vec{r})\times\langle G_N(\omega,\vec{r},1)\,G_N^*(\omega,1,2)\,G_N^*(\omega,2,\vec{r})\rangle\Psi(1)$$

$$+\Psi^*(\vec{r})\langle\Sigma^{(2)}(\omega,1)\rangle\langle G_N(\omega,\vec{r},1)\times G_N(\omega,1,2)G_N^*(\omega,2,\vec{r})\rangle\Psi(2)\}$$

$$-\frac{\alpha^2[\chi(\omega)]^2}{(\Psi|\Psi)}\int d^3r\int d1\,d2\,d3\ \Psi^*(\vec{r})\times\Psi(1)\Psi^*(2)\Psi(3)\langle G_N(\omega,\vec{r},1)\times G_N^*(\omega,1,2)G_N(\omega,2,3)G_N^*(\omega,3,\vec{r})\rangle \ . \quad (19)$$

We note that the last term on the right-hand side of (19) also occurs in the weak coupling calculations; it was evaluated by Maki[2] and later corrected by Caroli *et al.*[3] To evaluate the remaining terms, we begin with the impurity-averaged self-energy $\langle\Sigma^{(2)}(\omega,\vec{r})\rangle$. Going back to Eq. (4) we immediately see that $\langle G^{(2)}(\omega,\vec{r},\vec{r}')\rangle$ is negligibly small for $\vec{r} = \vec{r}'$, since the poles of the product of

the normal-state Green's functions lie on the same side of the real axis in the complex $i\omega$ plane. Thus inserting Eq. (3) into Eq. (5) we see that $\langle\Sigma^2\rangle$ has no longer to be determined by an integral equation, but is simply given by

$$\Sigma^{(2)}(\omega,\vec{r}) = +T\sum_{\omega'}\lambda(\omega,\omega')\alpha^2(\chi(\omega'))^2\int d1\,d2\;\Psi(1)$$

$$\times\langle G_N(\omega,\vec{r},1)\,G_N^*(\omega,1,2)\,G_N(\omega,2,\vec{r})\rangle\;\Psi^*(2). \quad (20)$$

Here we inserted Eq. (14) and took only the first term into account. To evaluate $R^{(3)}$, we therefore have to calculate the impurity average in various terms which contains three normal-state Green's functions. This can be done using standard techniques and we will write only the following final result:

$$\overline{R}^{(3)} = -\frac{4\pi N(0)\alpha^2\,\overline{|\Psi|^4}}{\overline{|\Psi|^2}} \times\left(\frac{[\chi(\omega)]^2}{[2|\omega Z(\omega)|+D\epsilon_0]^3}+\frac{|\omega|\,Z^{(2)}(\omega)}{[2|\omega Z(\omega)|+D\epsilon_0]^2}\right), \quad (21)$$

with

$$Z^{(2)}(\omega') = -\frac{2\pi N(0)T}{\omega'}\sum_{\omega}\lambda(\omega',\omega)\,\frac{\mathrm{sgn}\omega\,[\chi(\omega)]^2}{[2|\omega Z(\omega)|+D\epsilon_0]^2}\,. \quad (22)$$

Next we consider the supercurrent density $\vec{j}_s(r)$ which can be determined from Eq. (3) according to

$$\vec{j}_s(r) = (ie/m)\,T\sum_{\omega}(\vec{\delta}_{\vec{r}}-\vec{\delta}_{\vec{r}'}+2ieA(r)) \times[G(\omega,\vec{r},\vec{r}')-G_N(\omega,\vec{r},\vec{r}')]_{\vec{r}=\vec{r}'}\,. \quad (23)$$

Again it is easy to show that the contribution of $G^{(2)}(\omega,\vec{r},\vec{r}')$ to the current density is negligibly small. The remaining term which contributes to Eq. (23) was already calculated by Maki[2] in the weak coupling limit and can be taken over to the present calculation with only slight modifications. We then get

$$\vec{j}_s(\vec{r}) = B2ie(\tilde{\delta}_1-\tilde{\delta}_2^*)\,\Psi^*(2)\;\Psi(1)\,|_{1=2=\vec{r}}\,, \quad (24)$$

where use has been made of Eq. (14), and only the term proportional to α^2 has been written down.

The magnetization M is determined by Eqs. (17) and (24). Writing this quantity as

$$-4\pi M = (H_{c2}-H_e)/\beta_A[2\kappa_2^2(T)-1]\,, \quad (25)$$

where β_A depends only on the structure of the fluxoid lattice and is defined by

$$\beta_A = \overline{|\Delta|^4}/(\overline{|\Delta|^2})^2, \quad (26)$$

we get for the parameter $\kappa_2(T)$, using standard methods,

$$\kappa_2^2(T) = \left\{\sum_\omega\frac{[\chi(\omega)]^4}{[2|\omega Z(\omega)|+D\epsilon_0]^3}\Big/\,8\pi e^2D^2N(0)4\pi T\left(\sum_\omega\frac{[\chi(\omega)]^2}{[2|\omega Z(\omega)|+D\epsilon_0]^2}\right)^2\right\}[1+F(T)]\,. \quad (27)$$

The function $F(T)$ is defined by

$$F(T) = \frac{\sum_\omega|\omega|[\chi(\omega)]^2\overline{Z}^{(2)}(\omega)/[2|\omega Z(\omega)|+D\epsilon_0]^2}{\sum_\omega[\chi(\omega)]^4/[2|\omega Z(\omega)|+D\epsilon_0]^3}\,. \quad (28)$$

This final expression for $\kappa_2(T)$ contains the H_{c2} solution $\chi(\omega)$ which is defined as a solution of Eq. (13), thus showing that we are left with a rather complicated problem. But we will not go into the great numerical work which is necessary for a detailed calculation of both $\kappa_2(T)$ and $H_{c2}(T)$, and we will only give some limits for the strong coupling corrections to $\kappa_2(T)$.

III. DISCUSSION

Let us first consider the transition temperature where we can put $\epsilon_0=0$ in Eq. (27). The function $\chi(\omega)$ which enters into this equation clearly is, apart from an ω independent factor, identical with the order parameter for bulk material with no magnetic field applied, i.e., it can be written

$$\chi(\omega) = \phi_b(\omega)/\sqrt{(T_c-T)}\,\big|_{T=T_c}\,. \quad (29)$$

Therefore, we can express $\kappa_2(T_c)$ in terms of the quantities I_1, I_2, and I_1' introduced by EA, obtaining

$$\kappa_2^2(T_c) = (I_1+I_1')/4\pi e^2I_2^2\,. \quad (30)$$

But one can also derive $\kappa_1(T_c)$ directly from the EA paper to show that $\kappa_1(T_c)=\kappa_2(T_c)$.[10]

Let us now express $\kappa(T_c)$ in terms of measurable quantities. Introducing the correct renormalized $\kappa_{\mathrm{BCS}}(T)$ of the BCS theory [cf. EA and Eq. (36)], we get

$$\kappa^2(T_c) = \kappa_{\mathrm{BCS}}^2(T_c)\,[1+F(T_c)], \quad (31)$$

$$\text{with}\quad \kappa_{\mathrm{BCS}}^2(T_c) = \frac{1}{32\pi e^2(D^*)^2N^*}\,\frac{-\Psi^{(2)}(\tfrac{1}{2})}{[\Psi^{(1)}(\tfrac{1}{2})]^2} \quad (32)$$

and $$F(T_c)=\left.\frac{H_{c\,\mathrm{obs}}^2\Delta_{\mathrm{BCS}}^4}{H_{c\,\mathrm{BCS}}^2\Delta_{\mathrm{obs}}^4}\right|_{T\to T_c}-1\ . \tag{33}$$

For lead, we get $F(T_c)=-0.09$. In deriving this value, we have used the energy-gap data of Gasparovic *et al.*,[11] which yield $[\Delta_{\mathrm{obs}}^4(T)/\Delta_{\mathrm{BCS}}^4(T)]=2.08$ for $T\to T_c$, and the data of Decker *et al.*[12] for $H_{c\,\mathrm{obs}}(T)$. To derive $dH_{c\,\mathrm{BCS}}/dT$ at $T=T_c$, one has to multiply the reduced BCS slope of 1.736 with $1.764(6\gamma/\pi)^{1/2}$, where γ is the *measured* electronic specific heat. This gives $H_{c\,\mathrm{obs}}^2/H_{c\,\mathrm{BCS}}^2=1.89$ for $T\to T_c$.

Next we consider the general case, i.e., Eq. (27). The first term on the right-hand side of this equation can be simplified very much. First of all, it always seems to be a good approximation to replace the normal-state renormalization factor $Z(\omega)$ by a constant $Z(0)$, thus neglecting all pair breaking due to the electron-phonon interaction. This $Z(0)$ then renormalizes the density of states and the diffusion constant in exactly the right way; thus,

$$N(0)\to N^*=\tilde{Z}(0)N(0)\ , \tag{34}$$

$$D\to D^*=D/\tilde{Z}(0)\ .$$

Because of the good convergence of the ω sums in this first term, we may also replace $\chi(\omega)$ by a constant value. After these approximations, the first term on the right-hand side of Eq. (27) clearly goes over into the following BCS-like κ_2 expression[3] which we call $\bar{\kappa}_2$:

$$\bar{\kappa}_2^2(T)=\frac{1}{32\pi e^2(D^*)^2N^*}\,\frac{-\Psi^{(2)}(\frac{1}{2}+D^*\epsilon_0/4\pi T)}{\Psi^{(1)}(\frac{1}{2}+D^*\epsilon_0/4\pi T)}\ . \tag{35}$$

Therefore, we have

$$\kappa_2(T)=\bar{\kappa}_2(T)[1+F(T)]^{1/2}\ . \tag{36}$$

For $T=T_c$ and for $T=0$, $\bar{\kappa}_2$ is identical with the correct renormalized BCS κ_2. For intermediate temperatures, however, $\bar{\kappa}_2(T)$ is not equal to $\kappa_{2\,\mathrm{BCS}}(T)$, since the temperature dependence of $\bar{\kappa}_2(T)$ is given by the $H_{c2}(T)$ modified by strong coupling, i.e., by the *measured* $H_{c2}(T)$.

Now let us consider the function $F(T)$ which does not appear in a BCS-like calculation. Using Eq. (22) we rewrite this quantity as

$$F(T)=2\pi N(0)T{\sum_{\omega'}}'\frac{[\chi(\omega)]^2\,\mathrm{sgn}\omega[\lambda(\omega-\omega')-\lambda(\omega+\omega')]\,\mathrm{sgn}\omega'(\chi(\omega'))^2}{[2|\omega Z(\omega)|+D\epsilon_0]^2[2|\omega' Z(\omega')|+D\epsilon_0]^2}\Bigg/{\sum_{\omega'}}'\frac{[\chi(\omega)]^4}{[2|\omega Z(\omega)|+D\epsilon_0]^3}\ . \tag{37}$$

Here we have assumed $\chi(\omega)$ to be an even function of ω, since $\lambda(\omega,\omega')=\lambda(|\omega-\omega'|)$. The prime on $\sum$ denotes summation over positive frequencies. To get the order of magnitude of $F(T)$ at $T=0$, we replace λ by a step function, $\lambda(|\omega|)=-g\Theta(\omega_0-|\omega|)$, with an ω_0 of the order of the Debye temperature, and neglect the ω dependence of $\chi(\omega)$ and $Z(\omega)$, i.e., we replace these quantities by their values for small ω. Performing then the ω integration, we get

$$F(T=0)\approx-\frac{N(0)g}{Z(0)}\left[-\frac{x^2}{1+x}+\frac{x^2}{2}\,1+\left(\frac{1}{(1+x)^2}\right)\ln\frac{x+2}{x}\right], \tag{38}$$

where $$x=\alpha/\omega D,\quad \alpha=D^*\epsilon_0(T=0); \tag{39}$$

for small x, $F(T=0)$ is given by

$$F(0)\approx-[N(0)g/\tilde{Z}(0)]x^2[\ln(2/x)-1]\ . \tag{40}$$

The positive quantity $N(0)g$ is of the order of unity.

Since α is of the order of T_c, $F(0)$ is completely negligible for weak coupling superconductors ($T_c\ll\omega_D$). For strong coupling superconductors as lead alloys, $F(0)$ may be of the order of 10%, i.e., of the same order as $F(T_c)$.

The above calculation shows that the parameter $\kappa_2(T)$ is only slightly modified if the strong electron-phonon interaction is properly taken into account, at least in the extreme dirty limit. But we believe that this statement remains valid independently of the mean free path, as suggested by the experiments quoted in the Introduction.[7]

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$$I_1 + I_1' = \frac{1}{2\pi}\left(\frac{\partial H_c(T)}{\partial T}\right)^2_{T \to T_c}$$

instead of $I_1 - I_1'$ on the left-hand side of this equation. As a matter of fact, one gets

$$\left(\frac{H_{c2,\,\mathrm{obs}}}{H_{c2,\,\mathrm{BCS}}}\right)_{T \to T_c} = \left.\frac{H_{c,\mathrm{obs}}{}^2\,\Delta_{\mathrm{BCS}}{}^2}{H_{c,\mathrm{BCS}}{}^2\,\Delta_{\mathrm{obs}}{}^2}\right|_{T \to T_c} .$$

Using the numerical values quoted below Eq. (33) of the present paper, we get for lead $(H_{c2,obs}/H_{c2,BCS})_{T \to T_c} = 1.3$.
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Superconductive Properties of the Excitonic Insulator*

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Analysis of a simple model of the excitonic insulator shows that the ordered phase exhibits electrical superconductivity whenever the conduction-band mass differs from the valence-band mass. *Interband* scattering of electrons by the magnetic vector potential plays an essential role. States of finite electric persistent current are demonstrated explicitly. The excitonic insulator is a system where diagonal versus off-diagonal long-range order is a function of one's bookkeeping.

I. INTRODUCTION

It is commonly believed that the excitonic insulator,[1] a hypothetical many-particle cooperative thermodynamic phase involving valence-band holes and conduction-band electrons, has the electrical properties of an insulator.[2] In this paper, we show that this is not the case for a simple model; rather the model has the electrical properties of a superconductor. Our model consists of a single spherical valence band $(\hbar^2/2m_a)\,(k_F^2 - k^2)$ and a single spherical overlapping conduction band $(\hbar^2/2m_b)\,(k^2 - k_F^2)$, energies being measured relative to that of the Fermi surface $k = k_F$. We show that whenever $m_b \neq m_a$, there is superconductivity.[3] Crucial to this demonstration is the inclusion of both *intraband* scattering and *interband* scattering of electrons by the magnetic vector potential. The latter kind of scattering has not been considered in the past.

Note that a hole at $-\vec{k}$ in an otherwise filled valence band can be consistently thought of as an excitation of momentum $+\hbar\vec{k}$, mass $+m_a$, energy $(\hbar^2/2m_a)\,(k^2 - k_F^2)$, and electric charge $+e$ ($-e$ being the electronic charge). We designate by $c_{k\downarrow}^\dagger$, $c_{k\downarrow}$ the creation and destruction operators associated with such an excitation of momentum $\hbar\mathrm{k}$. (Thus $c_{k\downarrow}^\dagger$ removes an electron from one-electron state $-\vec{k}$ in the valence band.) We designate by $c_{k\uparrow}^\dagger$, $c_{k\uparrow}$ the creation and destruction operators associated with an electron of momentum $\hbar\vec{k}$ in the conduction band.

In the absence of external electric and magnetic fields, the Hamiltonian for the excitonic insulator is

$$H_0 = \sum_{k,\,\sigma} \epsilon_{k\sigma}\, c_{k\sigma}^\dagger\, c_{k\sigma} - \sum_{k,\,k'} V_{kk'}\, c_{-k'\downarrow}^\dagger\, c_{k'\uparrow}^\dagger\, c_{k\uparrow}\, c_{-k\downarrow} , \tag{1.1}$$

where

$$\epsilon_{k\sigma} = (\hbar^2/2m_\sigma)\,(k^2 - k_F^0), \quad m_\downarrow = m_a, \quad m_\uparrow = m_b . \tag{1.2}$$

$V_{kk'}$ is the matrix element for the attractive Coulomb scattering between holes and electrons. The ground-state wave function of the ordered phase has the form

$$\Psi_0 = \prod_k [(1 - h_k)^{1/2} + h_k^{1/2}\, c_{k\uparrow}^\dagger\, c_{-k\downarrow}^\dagger]\,\Phi_0, \tag{1.3}$$

Φ_0 being the state of filled valence band and empty conduction band. In the interest of simplicity, we